Midterm test - Solution sheet

EPFL-LPAP

Lausanne, November 7, 2024

Parameter set for calculation and estimations

At the end of 1999 the LEP collider at CERN stored electron-position beams with a total beam energy of 105 GeV at a total beam current (two beams) of 7 mA.

Table 1: LEP2 parameters in 1999.

Storage ring length	26.7 km
Length of main dipoles	5.75 m
Number of main dipoles	3392
Number of bunches/beam	4
Energy loss per turn	$\approx$ 3 GeV

Transverse Dynamics (12 marks)

1. In a storage ring, why do the particles not fall to the floor, even though they are often stored for a long time? (Assume a perfect vacuum and consider only the guiding magnets.) Name the specific element in the system that contributes to maintaining the particle's trajectory. (1 mark)

   Every object with mass experiences gravitational forces. In an accelerator, particle motion results from the combined effects of electromagnetic and gravitational forces. Gravity would naturally cause particles to fall toward the floor (or the bottom of the beam pipe).
   However, to maintain the particles on their nominal trajectory, an upward-deflecting force is needed to counteract gravity. This force is provided by vertically focusing quadrupoles, which create a restoring force to keep the particles centered in the beam path, preventing them from drifting downward due to gravity.

2. Calculate the work done by a static magnetic field on a moving particle, using the Lorentz force equation. (1 mark)

   For a particle with charge q moving at velocity $\vec{v}$ in a magnetic field $\vec{B}$, the magnetic force is:

   $$\vec{F}_B = q\,\vec{v} \times \vec{B}$$

   Since $\vec{F}_B$ is perpendicular to $\vec{v}$, the dot product $\vec{F}_B \cdot \vec{v}$ is zero. Therefore, the work done by the magnetic field is:

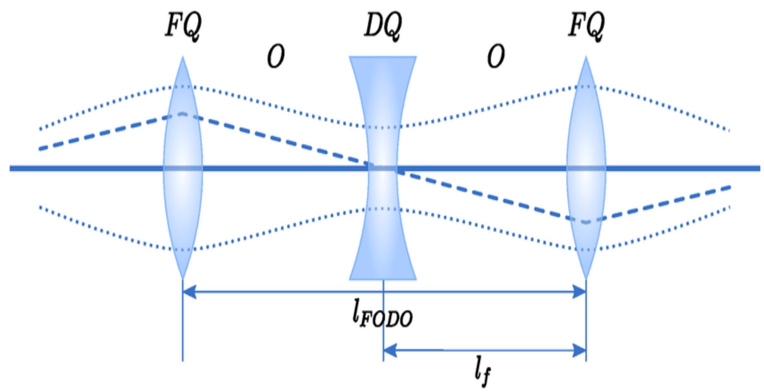
   $$W = \int \vec{F}_B \cdot d\vec{r} = 0$$

   Thus, a magnetic field does no work on a moving particle.

3. Using the LEP2 parameters above, compute the Lorentz factor γ, the magnetic field, and the revolution period. Use 0.511 MeV/c^2 for the rest mass of the electron/positron and $3 \cdot 10^8$ m/s for the speed of light c. (3 marks)

   $$\gamma = \frac{105 \cdot 10^9 eV}{511 \cdot 10^3 eV} \approx 205479.$$

   $$T = \frac{26.7\ km}{3 \cdot 10^8} = 89 \mu s$$



Total length dipoles: $L_d = 19504$ km, bending radius $\rho = 3104$ m

$$B \cdot \rho = 3.3356 \cdot 105\ GeV/c \quad \Longrightarrow B = 0.11T$$

4. In a typical focusing system, separated-function quadrupoles form structures known as FODO cells. Can you provide a drawing of a FODO cell and explain how overall focusing is achieved even when the focusing and defocusing quadrupoles have identical focal lengths? (2 marks)

   The focusing quadrupole focuses in one plane (e.g., the horizontal) while defocusing in the perpendicular plane (e.g., the vertical). The defocusing quadrupole does the opposite: it focuses in the vertical plane and defocuses in the horizontal. However, because the beam size is typically smaller in the defocusing quadrupole than in the focusing one, the focusing effect is stronger overall. This imbalance results in a net focusing effect over the entire FODO cell.

5. Write down the matrices for a focusing quadrupole, a defocusing quadrupole, and a drift space. Describe how you would compute the matrix representing motion through the entire FODO cell. Additionally, explain how to calculate the matrix for a complete ring composed entirely of identical FODO cells.(3 marks)

   The elements of a FODO cell (quadrupoles and drift spaces) are considered linear elements, allowing them to be described by matrices, whether using thin or thick lens approximations. By multiplying these matrices, we obtain a matrix that represents the entire FODO cell.

   1. Focusing Quadrupole:

   $$M_F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

   2. Defocusing Quadrupole:

   $$M_D = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

   3. Drift Space (length L):

   $$M_{\text{drift}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

   $$M_{\text{FODO}} = M_{\text{drift}} \cdot M_D \cdot M_{\text{drift}} \cdot M_F$$

   For a complete ring of identical FODO cells, multiply the FODO cell matrix by itself for the number of cells, resulting in the One-Turn Matrix.

6. Given the transport matrix (one plane) for a complete ring, how can you determine the tunes, as well as the beta, alpha, and gamma functions? (2 marks)

Using the transport matrix for a complete ring (one-turn matrix):

$$M_{1\text{-turn}} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

where μ is the phase advance per turn.

The tune is $Q = \mu/2\pi$ where the phase advance is given by the trace of the matrix, $\mathrm{Tr}(M_{1\text{-turn}}) = 2\cos\mu$. Once the tune is determined, the Twiss parameters can be extracted by identifying specific elements in the matrix:
- The β parameter can be determined from the element $M_{12} = \beta\sin\mu$,
- The γ parameter from the element $M_{21} = -\gamma\sin\mu$,
- The α parameter from $M_{11} - M_{22} = 2\alpha\sin\mu$.

This procedure can also be applied in the two-dimensional case.

Longitudinal Dynamics (10 marks)

1. Proton beams are injected at 14 GeV into the SPS. If the momentum compaction factor (α_c) is 3.086×10^{-4}, is the injection energy below or above transition? For this energy, what would be the minimum momentum compaction factor (α_c) acceptable to avoid crossing transition? (2 marks)

   The slip factor $\eta = \alpha_c - 1/\gamma^2$ is zero at transition. Therefore:

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha_c}} = 18$$

   The SPS gamma at injection $\gamma_{inj} = E/E_0 \approx 15$. Since $\gamma_{inj} < \gamma_{tr}$, the SPS injection energy is below transition. In addition, η must positive above transition, and therefore:

$$\alpha_c > \frac{1}{\gamma_{inj}^2} \approx 67 \times 10^{-3}$$

2. Considering the LHC, with proton beams injected at 450 GeV, with a circumference of 26659 m, and a RF frequency of 400.8 MHz, what is the maximum number of bunches in the ring? What is this number commonly called? (consider $c = 299\,792\,458\,\mathrm{m\,s^{-1}}$) (2 marks)

   In a synchrotron machine, the RF frequency is a multiple of the revolution frequency:

$$f_{RF} = h \cdot f_{rev} \tag{1}$$

   Where h is the maximum number of bunches in the ring, commonly known as harmonic number. The revolution frequency can be easily calculated through $f_{rev} = \frac{\beta c}{C}$. From the γ_{inj} calculated above we know that $\beta_{inj} = 0.9999978$. Therefore:

$$h = \frac{f_{RF}}{\beta_{inj} c/C} = 35641 \tag{2}$$

3. The LHC collision, center-of-mass energy is 14 TeV, therefore the energy gain from injection to collision is extremely high. Do we also expect a large change in the RF frequency? (2 marks)

   The beam is ultrarelativistic already at injection, meaning $\beta \approx 1$. Therefore, the change in the RF frequency will be negligible.
   The sudents might optionally calculate the exact RF frequency variation:
   From the exercise above we know that $f_{rev} = \frac{\beta c}{C}$, and $f_{RF} = h \cdot f_{rev}$. Therefore, as the beam energy increases, even if the variation of β is slight, f_{RF} will change too.

$$\beta_{inj} = 0.9999978,\ \beta_{col} = 0.999999991022. \tag{3}$$

In both injection and collision $\beta \approx 1$, thus we can conclude that the variation of f_{RF} at LHC extremely small:

$$\Delta f_{RF} = h \cdot \frac{c \cdot (\beta_{col} - \beta_{inj})}{C} = 0.878\,\text{kHz} \tag{4}$$

4. If the RF system is used to focus the beam longitudinally but not for acceleration, what should be the synchronous phase Φs? What Φs would you choose for maximum acceleration? (2 marks)

   Since we are operating above transition, and assuming $V = V_0\ sin(\Phi_s)$:

   - $\phi_s = \pi$ for longitudinal focusing with no acceleration,
   - $\phi_s = \pi/2$ for maximum acceleration and
   - ϕ_s = -$\pi/2$ (or $3\pi/2$) for maximum deceleration.

5. Around 65.62% of the LHC circumference is occupied by dipoles. What is the dipole field at injection? And just before the collision? If the ramping time of such magnets is around 9.7 minutes, calculate the energy gain per turn (assume $\beta \approx 1$). (2 marks)

   The bending radius ρ can be easily obtained:

$$\rho = \frac{0.6562 \cdot C}{2\pi} = 2784.2\,\text{m} \tag{5}$$

   the required magnetic field is thus calculated through the magnetic rigidity formula:

$$B \cdot \rho\ [\text{T}\cdot\ \text{m}] = 3.3356 \cdot p\ [\text{GeV/c}] \longrightarrow B\ [\text{T}] = 3.3356 \cdot \frac{p}{\rho}\ [\text{GeV/c} \cdot \text{m}^{-1}] \tag{6}$$

   and considering p = 450 GeV/c at injection and p = 7000 GeV/c at collision (p $\approx$ E):

   - B = 0.54 T at injection and
   - B = 8.38 T right before collision.

   Finally, we can consider a constant revolution period of the beam assuming $\beta \approx 1$.

$$\tau_{rev} = \frac{C}{\beta c} \approx \frac{C}{c} = 88.9\,\mu\text{s} \tag{7}$$

   We can interpret the ramping time as the total time the beams are stored in the LHC before collision. Therefore:

$$\Delta E_{turn} = \frac{(E_{col} - E_{inj}) \cdot \tau_{rev}}{t_{ramping}} = \frac{(7000\ \text{GeV} - 450\ \text{GeV}) \cdot 88.9\,\mu\text{s}}{5.82 \times 10^8 \mu\text{s}} \approx 1\ \text{MeV} \tag{8}$$

Synchrotron Radiation and Electron Dynamics, Radiation Damping (8 marks)

1. Explain what determines the beam size of electron beams in the horizontal plane in circular accelerators. Why is the vertical beam size of electron beams typically much smaller than the horizontal beam size? Why does this difference not occur in proton accelerators? (3 marks)

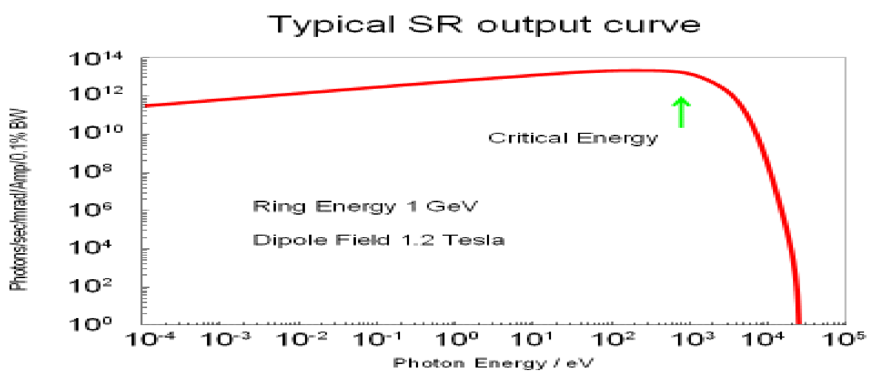
   The electron beam profile is significantly influenced by synchrotron radiation. In the bending plane, synchrotron radiation increases the emittance, while in the perpendicular plane, radiation damping reduces it. This creates a large difference in emittance ratios between the two planes.

   In contrast, protons are minimally affected by synchrotron radiation, so their emittances are largely determined by the history of the particles from the source to the measurement point. As a result, proton beam profiles tend to be round unless shaped otherwise intentionally.

2. Could the beam size of electron beams be damped to zero emittance? (1 marks)

   In a classical model, the emittance could, in theory, be damped to zero. However, because synchrotron radiation is a statistical process and emitted in discrete quanta, complete damping to zero is not possible.

3. Make a drawing of a Synchrotron Radiation Spectrum and define the Critical Photon Energy. How does the Critical Energy depend on the bending radius and the beam energy? (2 marks)



   The critical energy (or frequency) is defined as the energy where the integrated spectral energy distribution is separated into two equal parts.

4. Synchrotron Radiation is more critical for particles with a small mass at similar energies, why? (2 marks)

   The radiation effects strongly depend on γ, e.g. the power loss is $\propto \gamma^4$. Therefore it is most relevant for ultra-relativistic particles. Important consequences are emittance increase and radiation damping for light particles and high energies.

Bonus

1. Why are Cyclotrons not used to accelerate electrons ? Give one reason. (2 marks)

   The synchronicity condition requires an increase of the speed with energy, i.e. non-relativistic particles. Almost never true for electrons.